Rings with the Beachy-Blair condition *

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Abstract

A ring satisfies the left Beachy-Blair condition if each of its faithful left ideal is cofaithful. Every left zip ring satisfies the left Beachy-Blair condition, but both properties are not equivalent. In this paper we will study the similarities and the differences between zip rings and rings with the Beachy-Blair condition. We will also study the relationship between the Beachy-Blair condition of a ring and its skew polynomial and skew power series extensions. We give an example of a right zip ring that is not left zip, proving that the zip property is not symmetric.

Key words: Zip rings, rings with the Beachy-Blair condition, Armendariz rings.
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1 Introduction

The first time that the concept of zip ring appeared as it is known nowadays was in 1989, by Faith in [4]. Previously, Beachy and Blair in [1] (1975) and Zelmanowitz in [8] (1976), introduced a more general property. In [1], Beachy and Blair defined rings whose faithful left ideals are cofaithful (we will call these rings, rings with the left Beachy-Blair condition) and, in [8], Zelmanowitz worked with rings with the "finite intersection property" on annihilator left ideals. Both properties are equivalent, but they were introduced independently and parallelly, obtaining quite different results.

Zelmanowitz, in [8], noted that his condition was less restrictive than DCC$\perp$ (descending chain condition on annihilators), that is, there exist rings with the left Beachy-Blair condition that do not satisfy the left DCC$\perp$, but every ring satisfying the left DCC$\perp$ satisfies the left Beachy-Blair condition. In fact, the reason why Zelmanowitz introduced his property was to weaken the chain condition on annihilators. From the point of view of Beachy and Blair, the Beachy-Blair condition arose in order to give a characterization of semiprime left Goldie rings. They proved in [1] that a semiprime ring is left Goldie (that is, it satisfies the ascending chain condition on left annihilators and it has finite

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uniform dimension) if and only if it satisfies the left Beachy-Blair condition and that every nonzero left ideal contains a nonzero uniform left ideal.

Let us recall some basic definitions.

***Definition* 1.1**

A ring R is left zip if for every subset $X \subseteq R$ such that $l.ann_R(X) = \{0\}$, there exists a finite subset $F \subseteq X$ such that $l.ann_R(F) = \{0\}$, where $l.ann_R(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$ denotes the left annihilator of X in R.

Analogously we can define right zip ring. A ring is zip if it is both left and right zip.

***Definition* 1.2**

A ring R satisfies the left Beachy-Blair condition if for every faithful left ideal I of R (that is, $l.ann_R(I) = \{0\}$), there exists a finite subset $F \subseteq I$ such that $l.ann_R(F) = \{0\}$.

Analogously we can define the right Beachy-Blair condition. A ring satisfies the Beachy-Blair condition if it satisfies both the left and the right Beachy-Blair conditions.

Throughout this paper, all rings are supposed to be associative with identity. Unless otherwise stated, all results are given on the left but are also true on the right.

Every left zip ring satisfies the left Beachy-Blair condition, but there are examples of rings with the left Beachy-Blair condition that are not left zip ($R[[x]]$ in Proposition 4.4 is an example of this kind of rings). However, for commutative or reduced rings, it is not difficult to see that both properties are equivalent. In general, we have the following:

$$\text{Left DCC}\perp \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{Left Zip} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{Left Beachy-Blair condition.}$$

Faith, in [5], proposed the following questions regarding zip rings.

Let R be any ring.

1. Does R being a left zip ring imply $R[x]$ being left zip?
2. Does R being a left zip ring imply $M_n(R)$ being left zip?
3. Does R being a left zip ring imply $R[G]$ being left zip when G is a finite group?

Cedó in [2] (1991) answered all these questions in the negative. However, when R is a commutative ring, Beachy and Blair ([1, Proposition 1.9]) gave a positive answer to 1. and Cedó ([2, Proposition 1]) gave a positive answer to 2. In [5], Faith proved that if R is a commutative zip ring and G is a finite abelian group, then the group ring $R[G]$ is zip.

It is natural then to ask these same questions for rings with the left Beachy-Blair condition.

Beachy and Blair proved that the Beachy-Blair condition is Morita invariant ([1, Corollary 1.2]), and, therefore, a ring R satisfies the left Beachy-Blair condition if and only if $M_n(R)$ satisfies the left Beachy-Blair condition for all $n \geq 1$, so the answer to 2. is positive for rings with the left Beachy-Blair condition, even in the noncommutative case.

Note that the example of Cedó of a domain S such that $M_n(S)$ is not right zip ([2, Example 1]), gives us an example of a ring, $M_n(S)$, that satisfies the right Beachy-Blair condition, since S satisfies the right Beachy-Blair condition, but is not right zip.

It is widely believed that the answer to 1. for rings with the left Beachy-Blair condition should be negative in general, but it remains as an open problem, since no counterexample has been found so far. However, in Section 2 we will prove that, under certain conditions, the answer to 1. is positive for rings with the left Beachy-Blair condition. In Section 3 we will study the relationship between the Beachy-Blair condition of a ring and its skew power series extension, and compare our results with similar known results for zip rings. Finally, in Section 4, we will construct an example that answers in the negative some open problems regarding the Beachy-Blair condition and the zip property.

2 Skew polynomial extensions over rings with the Beachy-Blair condition

In this section we will study the relationship between the Beachy-Blair condition of a ring and its skew polynomial extension. Let R be a ring and α be an endomorphism of R. The α-skew polynomial extension of R, denoted by $R[x;\alpha]$, is the ring with elements of the form $\sum_{i=0}^{n} a_i x^i$, with $a_i \in R$, and with the multiplication defined by

$$(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} b_j x^j) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i \alpha^i(b_j) x^{i+j}$$

and the sum defined by

$$(\sum_{i=0}^{n} a_i x^i) + (\sum_{j=0}^{m} b_j x^j) = \sum_{i=0}^{\max\{n,m\}} (a_i + b_i) x^i.$$

In particular, $xb = \alpha(b)x$ for all $b \in R$.

***Definition* 2.1**
Let R be a ring and α be an endomorphism of R. R is α-skew Armendariz if for all $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ such that $f(x)g(x) = 0$ we

have that $a_i\alpha^i(b_j)=0$ for all i,j. When α is the identity of R, we say that R is an Armendariz ring.

Let α be an automorphism of R. Let $\Gamma=\{l.ann_R(U)\mid U\subseteq R\}$ and $\Delta=\{l.ann_{R[x;\alpha]}(V)\mid V\subseteq R[x;\alpha]\}$. Since for all $U\subseteq R$ we have that $l.ann_{R[x;\alpha]}(U)=R[x;\alpha]l.ann_R(U)$, we can define the map $\phi:\Gamma\to\Delta$ by $\phi(A)=R[x;\alpha]A$, for all $A\in\Gamma$.

On the other hand, for all $V\subseteq R[x;\alpha]$ we define $C_V\subseteq R$ by $C_V=\bigcup_{f(x)\in V} C_f$ and $C_f=\{a_0,a_1,\ldots,a_n\}\cup\{0\}$, where $f(x)=\sum_{i=0}^{n}a_ix^i$. Then, since $l.ann_{R[x;\alpha]}(V)\cap R=l.ann_R(C_V)$, we can define the map $\psi:\Delta\to\Gamma$ by $\psi(B)=B\cap R$ for all $B\in\Delta$.

It is easy to see that ϕ is always injective and that ψ is always surjective. Moreover, Cortes, in [3], noted that ϕ is bijective if and only if ψ is bijective, and, in this case, one is the inverse of the other, and proved, similarly as Hirano did in [7] for polynomial rings, that this happens if and only if the ring R is α-skew Armendariz ([3, Lemma 2.7]).

Note that neither the zip property nor the Beachy-Blair condition pass to subrings in general. However, Cortes in [3] proved that if $R[x;\alpha]$, where α is an automorphism of a ring R, is left zip, then the ring R is left zip as well. In the following Lemma we will see that, although we can't prove the same for the Beachy-Blair condition, adding another assumption, a similar result can be proven.

Lemma 2.2

Let R be a ring and α be an automorphism of R. Then, if $R[x;\alpha]$ satisfies the left Beachy-Blair condition and R is α-compatible (i.e. for all $a,b\in R$, $ab=0$ if and only if $a\alpha(b)=0$), then R satisfies the left Beachy-Blair condition.

Proof. Let I be a left ideal of R such that $l.ann_R(I)=\{0\}$. Then $l.ann_{R[x;\alpha]}(I)=\phi(l.ann_R(I))=\phi(\{0\})=\{0\}$. Moreover,

$$l.ann_{R[x;\alpha]}(R[x;\alpha]I)\subseteq l.ann_{R[x;\alpha]}(I)=\{0\}$$

and $R[x;\alpha]I$ is a left ideal of $R[x;\alpha]$. Then, since $R[x;\alpha]$ satisfies the left Beachy-Blair condition, there exists a finite subset $Y\subseteq R[x;\alpha]I$ such that $l.ann_{R[x;\alpha]}(Y)=\{0\}$. Let $X=C_Y$ (the subset containing all the coefficients of all polynomials in Y). Let $a\in l.ann_R(X)$. Then, for all $f(x)\in Y$, we have that $af(x)=0$, so $a\in l.ann_{R[x;\alpha]}(Y)=0$. Therefore, $l.ann_R(X)=\{0\}$.

Note that, since I is a left ideal of R, every $g(x)\in R[x;\alpha]I$ is of the form $g(x)=\sum_{i=0}^{n}\alpha^i(a_i)x^i$, where $a_i\in I$. Assume $X=\{\alpha^{i_1}(a_1),\ldots,\alpha^{i_m}(a_m)\}$ for some $X'=\{a_1,\ldots,a_m\}\subseteq I$. If $a\in l.ann_R(X')$ then, since R is α-compatible, $a\alpha^{i_k}(a_k)=0$ for all $1\leq k\leq m$, so $a\in l.ann_R(X)=\{0\}$. Therefore, $l.ann_R(X')=\{0\}$ and R satisfies the left Beachy-Blair condition. ∎

Cortes in [3, Theorem 2.8] proved that, when the ring R is α-skew Armendariz, R is left zip if and only if $R[x;\alpha]$ is left zip. A similar result is also true for rings with the Beachy-Blair condition.

Theorem 2.3

Let R be an α-skew Armendariz ring with α an automorphism of R.

If R satisfies the left Beachy-Blair condition then $R[x;\alpha]$ satisfies the left Beachy-Blair condition. Moreover, if R is α-compatible, then the converse is true.

Proof. We will denote by S the skew polynomial ring $S = R[x;\alpha]$. Suppose that R satisfies the left Beachy-Blair condition. Let $J \subseteq S$ be a left ideal of S such that $l.ann_S(J) = \{0\}$. Then, if $J' = JS$ is the (two-sided) ideal of S generated by J, we have that $l.ann_S(J') = \{0\}$. Then, $l.ann_R(C_{J'}) = \psi(l.ann_S(J')) = \psi(\{0\}) = \{0\}$ and $C_{J'} \subseteq R$. We shall see that $C_{J'}$ is an ideal of R.

(1) If $s \in C_{J'}$, then there exists $f(x) \in J'$ such that $f(x) = \sum_{i=0}^{n} a_i x^i$ and $s = a_i$ for some i. Let $r \in R$. Since J' is an ideal of S, $g(x) = f(x)\alpha^{-i}(r), h(x) = rf(x) \in J'$, and the coefficients of x^i in $g(x)$ and $h(x)$ are $a_i r$ and ra_i respectively. Therefore $sr, rs \in C_{J'}$.

(2) If $r_1, r_2 \in C_{J'}$, there exist $f(x), g(x) \in J'$ such that $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ and $r_1 = a_i$ for some i, $r_2 = b_j$ for some j. We want to see that $r_1 + r_2 \in C_{J'}$. Assume without loss of generality that $i \leq j$. Since $f(x), g(x) \in J'$ and J' is an ideal of S we have that $h(x) = f(x)x^{j-i} + g(x) \in J'$, and the coefficient of x^j in $h(x)$ is $a_i + b_j$, so $r_1 + r_2 \in C_{J'}$.

Now, since R satisfies the left Beachy-Blair condition, there exists a finite subset $X \subseteq C_{J'}$ such that $l.ann_R(X) = \{0\}$. Assume that $X = \{a_1, \ldots, a_n\}$, then, for every a_i there exists a polynomial $f_i(x)$ in J' such that $a_i \in C_{f_i}$. Let $Y = \{f_1(x), \ldots, f_n(x)\} \subseteq J'$. Clearly $X \subseteq C_Y$, so $l.ann_R(C_Y) \subseteq l.ann_R(X) = \{0\}$. Now, $l.ann_R(C_Y) = \psi(l.ann_S(Y)) = \{0\}$, and, since R is α-skew Armendariz, by [3, Lemma 2.7], ψ is bijective, so $l.ann_S(Y) = \{0\}$.

By the definition of J', there exist integers $m_1, \ldots m_n \geq 0$, and polynomials $f_{i,j}(x) \in J$, $s_{i,j}(x) \in S$ for all $1 \leq i \leq n$, $0 \leq j \leq m_i$ such that

$$f_i(x) = \sum_{j=0}^{m_i} f_{i,j}(x) s_{i,j}(x).$$

Let $Y' = \{f_{i,j}(x) \mid 1 \leq i \leq n \text{ and } 0 \leq j \leq m_i\} \subseteq J$. Clearly, $l.ann_S(Y') \subseteq l.ann_S(Y) = \{0\}$. Therefore, S satisfies the left Beachy-Blair condition.

If R is α-compatible, then the converse follows by Lemma 2.2. ∎

An easy consequence of Theorem 2.3 is the following result.

Corollary 2.4

Let R be an Armendariz ring. Then, R satisfies the left Beachy-Blair condition if and only if $R[x]$ satisfies the left Beachy-Blair condition.

Although in 1991 Cedó proved that there exist right zip rings R such that $R[x]$ is not right zip, it is still an open problem whether there exists such an example for the Beachy-Blair condition or not.

3 Skew power series extensions over rings with the Beachy-Blair condition

In this section we will study the relationship between the Beachy-Blair condition of a ring and its skew power series extension. Let R be a ring and α be an endomorphism of R. The α-skew power series extension of R, denoted by $R[[x;\alpha]]$, is the ring with elements of the form $\sum_{i\geq 0} a_i x^i$, with $a_i \in R$, and with the multiplication defined by

$$(\sum_{i\geq 0} a_i x^i)(\sum_{j\geq 0} b_j x^j) = \sum_{i\geq 0}\sum_{j\geq 0} a_i \alpha^i(b_j) x^{i+j}$$

and the sum defined by

$$(\sum_{i\geq 0} a_i x^i) + (\sum_{j\geq 0} b_j x^j) = \sum_{i\geq 0} (a_i + b_i) x^i.$$

In particular, $xb = \alpha(b)x$ for all $b \in R$.

First of all, it is important to remind that it remains as an open problem whether or not the Beachy-Blair condition passes to the power series ring in general. For zip rings, Cedó in [2, Example 2], proved that, for any field $\mathbb{K}$, there exists a right zip $\mathbb{K}$-algebra R such that $R[x]$ is not right zip. We will prove that this example of Cedó also satisfies that $R[[x]]$ is not right zip.

Example 3.1

For any field $\mathbb{K}$, there exists a right zip $\mathbb{K}$-algebra R such that $R[[x]]$ is not right zip.

Proof. Recall the construction of the example of Cedó [2, Example 2].

Let $\mathbb{K}$ be a field. Let R be the $\mathbb{K}$-algebra with set of generators $A = \{a_\infty, a_\lambda, a_{0,n}, a_{1,n}, b_{1,n}, b_{2,n} \mid n \geq 0, \lambda \in \mathbb{K}\}$ and with relations:

(i) $a_{0,i}b_{1,j} = a_{0,i}b_{2,j} = a_{1,i}b_{1,j}$ for all $j \geq i \geq 0$,

(ii) $a_{1,i}b_{2,j} = 0$ for all $j \geq i \geq 0$,

(iii) $a_{1,i}a_\infty = (a_{0,i} + \lambda a_{1,i})a_\lambda = 0$ for all $i \geq 0$ and for all $\lambda \in \mathbb{K}$,

(iv) $a_\infty x = a_\lambda x = b_{k,j}x = 0$ for all $j \geq 0$, for all $\lambda \in \mathbb{K}$, for $k \in \{1,2\}$ and for all $x \in A$.

It is not hard to verify that the set U of all the products of the form:

1. $a_{l_1,i_1} \cdots a_{l_n,i_n}$ with $n \geq 0$ and $l_\nu \in \{0,1\}$, $i_\nu \geq 0$ for all $\nu \in \{1,\dots,n\}$,
2. $a_{l_1,i_1} \cdots a_{l_n,i_n} a_\mu$ with $n \geq 0$, $l_\nu \in \{0,1\}$, $i_\nu \geq 0$ for all $\nu \in \{1,\dots,n\}$, $l_n = 0$ and $\mu \in \mathbb{K} \cup \{\infty\}$,
3. $a_{l_1,i_1} \cdots a_{l_n,i_n} a_0$ with $n > 0$, $l_\nu \in \{0,1\}$, $i_\nu \geq 0$ for all $\nu \in \{1,\dots,n\}$ and $l_n = 1$,
4. $a_{l_1,i_1} \cdots a_{l_n,i_n} b_{k,j}$ with $n \geq 0$, $l_\nu \in \{0,1\}$, $i_\nu, j \geq 0$ for all $\nu \in \{1,\dots,n\}$, $k \in \{1,2\}$ and if $n > 0$ and $j \geq i_n$ then $l_n = 0$ and $k = 1$,

is a $\mathbb{K}$-basis for R.

Let $\alpha \in R$, then $\alpha = \sum_{u \in U} \alpha(u)u$, where $\alpha(u) \in \mathbb{K}$ and $\alpha(u) = 0$ for almost all $u \in U$. We define the support of α, $Supp(\alpha)$, to be $Supp(\alpha) = \{u \in U \mid \alpha(u) \neq 0\}$.

Cedó in [2, Example 2] proved that R is right zip. We shall see that $S = R[[x]]$ is not right zip. Let $X = \{a_{0,i} - a_{1,i}x \mid i \geq 0\}$ and $X_n = \{a_{0,i} - a_{1,i}x \mid n \geq i \geq 0\}$. It is easy to see that, for all $n \geq 0$, $b_{1,n} - b_{2,n} + b_{1,n}x + b_{2,n}x^2 \in r.ann_S(X_n)$, so for every finite subset F of X we have that $r.ann_S(F) \neq 0$. Let us see now that $r.ann_S(X) = \{0\}$.

Suppose that $r.ann_S(X) \neq \{0\}$. Then, there exists $\alpha = \sum_{i \geq 0} \alpha_i x^i \in r.ann_S(X)$ such that $\alpha_0 \neq 0$. Since $a_{0,i}\alpha_0 = 0$ for all $i \geq 0$, we have that $\alpha_0 = \alpha_0(a_0)a_0$ (see the proof of [2, Example 2] for details). Now, $a_{0,i}\alpha_1 = \alpha_0(a_0)a_{1,i}a_0$, but $a_{1,i}a_0 \notin Supp(a_{0,i}\alpha_1)$, which is a contradiction. Therefore, $r.ann_S(X) = \{0\}$, so S is not right zip. ∎

Note that in Example 3.1, both $R[x]$ and $R[[x]]$ satisfy the right Beachy-Blair condition. Moreover, in the rest of this section, we will see that, under certain conditions, the Beachy-Blair condition passes to power series extensions.

***Definition* 3.2**

Let R be a ring and α be an endomorphism of R. We say that R is strongly α-skew Armendariz if for all $f(x) = \sum_{i \geq 0} a_i x^i$, $g(x) = \sum_{j \geq 0} b_j x^j \in R[[x;\alpha]]$ such that $f(x)g(x) = 0$ we have that $a_i\alpha^i(b_j) = 0$ for all $i, j \geq 0$. When α denotes the identity of R, we say that R is strongly Armendariz.

It is clear that if a ring is strongly α-skew Armendariz, then it is α-skew Armendariz, but the converse is not true in general.

Example 3.3

There exists an Armendariz ring which is not strongly Armendariz.

Proof. Let $\mathbb{K} = \mathbb{Z}_2$ and let R be the $\mathbb{K}$-algebra presented with generators $\{a_i, b_j \mid i, j \geq 0\}$ and with relations

(a) $a_i b_0 = a_{i-1}b_1 + a_{i-2}b_2 + \cdots + a_0 b_i$, for all $i \geq 1$,

(b) $a_0 b_0 = b_i a_j = a_i a_j = b_i b_j = 0$, for all $i, j \geq 0$.

Let $B_a = \{a_i \mid i \geq 0\}$, $B_b = \{b_j \mid j \geq 0\}$ and $B_2 = \{a_i b_j \mid a_i b_j \mid i \geq 0, j \geq 1\}$, then it is easy to check that $B = \{1\} \cup B_a \cup B_b \cup B_2$ is a $\mathbb{K}$-basis for R. For all $r \in R$, $r = \sum_{z \in B} r(z)z$, where $r(z) \in \mathbb{K}$ and $r(z) = 0$ for almost all $z \in B$. We define the support of r, $Supp(r)$, to be $Supp(r) = \{z \in B \mid r(z) \neq 0\}$.

If we denote by U_i the set of all finite sums of elements of B_i, with $i \in \{a, b, 2\}$, then $R = U_0 \oplus U_a \oplus U_b \oplus U_2$, with $U_0 = \mathbb{K}$, and every element $r \in R$ can be written as $r = r_0 + r_a + r_b + r_2$ with $r_i \in U_i$.

In order to continue the proof, we need the following technical lemma.

Lemma 3.4
Let $f(x) = \sum_{i=0}^{n} r_i x^i$, $g(x) = \sum_{j=0}^{m} s_j x^j \in R[x] \setminus \{0\}$ *be such that* $f(x)g(x) = 0$. *Assume that* $r_i = r_{i,0} + r_{i,a} + r_{i,b} + r_{i,2}$ *and* $s_j = s_{j,0} + s_{j,a} + s_{j,b} + s_{j,2}$ *with* $r_{i,k}, s_{j,k} \in U_k$ *for all* $i, j \geq 0$ *and for all* $k \in \{0, a, b, 2\}$. *Then,* $r_{i,0} = s_{j,0} = 0$ *for all* $i, j \geq 0$.

Proof of the Lemma. We have that $f(x)g(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} r_i s_j \; x^{i+j} = 0$. Then:

(1) $\sum_{i=0}^{n} \sum_{j=0}^{m} r_{i,0} s_{j,0} \; x^{i+j} = 0$

(2) $\sum_{i=0}^{n} \sum_{j=0}^{m} (r_{i,0} s_{j,a} + r_{i,a} s_{j,0}) \; x^{i+j} = 0$

(3) $\sum_{i=0}^{n} \sum_{j=0}^{m} (r_{i,0} s_{j,b} + r_{i,b} s_{j,0}) \; x^{i+j} = 0$

(4) $\sum_{i=0}^{n} \sum_{j=0}^{m} (r_{i,0} s_{j,2} + r_{i,a} s_{j,b} + r_{i,2} s_{j,0}) \; x^{i+j} = 0$

Assume that there exists some i such that $r_{i,0} \neq 0$ and i_1 is the minimum with this property. If there exists j such that $s_{j,0} \neq 0$, assuming that j_1 is minimum with this property, then, by (1), we have that $r_{i_1+j_1,0} s_{0,0} + \cdots + r_{i_1,0} s_{j_1,0} + \cdots + r_{0,0} s_{i_1+j_1,0} = r_{i_1,0} s_{j_1,0} = 0$, but this is a contradiction by the definition of i_1 and j_1. Therefore, $s_{j,0} = 0$ for all $j \geq 0$.

Now we have that:

(2) $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} r_{i,0}s_{j,a}\ x^{i+j} = 0$

(3) $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} r_{i,0}s_{j,b}\ x^{i+j} = 0$

(4) $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} (r_{i,0}s_{j,2} + r_{i,a}s_{j,b})\ x^{i+j} = 0$

Assume that there exists j_2 such that $s_{j_2,a} \neq 0$ and j_2 is the minimum with this property. Then, by (2), $r_{0,0}s_{i_1+j_2,a} + \cdots + r_{i_1,0}s_{j_2,a} + \cdots + r_{i_1+j_2,0}s_{0,a} = r_{i_1,0}s_{j_2,a} = s_{j_2,a} = 0$, but this is a contradiction by the definition of i_1 and j_2. Therefore, $s_{j,a} = 0$ for all $j \geq 0$. Analogously, by (3), we have that $s_{j,b} = 0$ for all $j \geq 0$.

Now we have that:

(4) $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} r_{i,0}s_{j,2}\ x^{i+j} = 0$

and, similarly as above, if there exists $j_3 \geq 0$ such that $s_{j_3,2} \neq 0$ and j_3 is the minimum with this property. Then, by (4), $r_{0,0}s_{i_1+j_3,2} + \cdots + r_{i_1,0}s_{j_3,2} + \cdots + r_{i_1+j_3,0}s_{0,2} = r_{i_1,0}s_{j_3,2} = s_{j_3,2} = 0$, but this is a contradiction by the definition of i_1 and j_3. Therefore, $s_{j,2} = 0$ for all $j \geq 0$, and so, $g(x) = 0$, but this is a contradiction. Then, $r_{i,0} = 0$ for all $i \geq 0$. Similarly we can see that $s_{j,0} = 0$ for all $j \geq 0$. This completes the proof of the Lemma. ▮

Now we continue the proof of Example 3.3.

Define $f'(x) = \displaystyle\sum_{i\geq 0} a_i x^i$ and $g'(x) = \displaystyle\sum_{j\geq 0} b_j x^j \in R[[x]]$. By (a), it is clear that $f'(x)g'(x) = 0$. However, $a_1b_0 \neq 0$, and then, R is not strongly Armendariz.

We shall see that R is Armendariz. Let $f(x), g(x) \in R[x] \setminus \{0\}$ be such that $f(x)g(x) = 0$. Assume that $f(x) = \displaystyle\sum_{i=0}^{n} r_i x^i$ and $g(x) = \displaystyle\sum_{j=0}^{m} s_j x^j$. We want to see that $r_is_j = 0$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$. Assume that $r_i = r_{i,0} + r_{i,a} + r_{i,b} + r_{i,2}$ and $s_j = s_{j,0} + s_{j,a} + s_{j,b} + s_{j,2}$, with $r_{i,k}, s_{j,k} \in U_k$ for all $i, j \geq 0$ and for all $k \in \{0, a, b, 2\}$.

By Lemma 3.4, we may assume that $r_{i,0} = s_{j,0} = 0$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$. Then, $r_is_j = r_{i,a}s_{j,b}$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$, so we may assume that $r_i \in U_a$ and $s_j \in U_b$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$. Without loss of generality we may also assume that $r_0, r_n, s_0, s_m \neq 0$.

We define the length of an element $r \in R \setminus \{0\}$ by

$$l(r) = \max\{l(u) \mid u \in Supp(r)\}$$

where $\left\{\begin{array}{lcl} l(a_i) & = & i \\ l(b_j) & = & j \\ l(a_ib_j) & = & i+j \\ l(1) & = & -1 \end{array}\right.$,

and $l(0) = -\infty$.

We define the map δ over elements $r_a \in U_a \setminus \{0\}$, $r_b \in U_b \setminus \{0\}$ and $r_2 \in U_2\setminus\{0\}$, by $\delta(r_a) = a_{l(r_a)}$, $\delta(r_b) = b_{l(r_b)}$ and $\delta(r_2) = a_i b_j$, where $a_i b_j \in Supp(r_2)$ and j is the biggest satisfying $i + j = l(r_2)$.

We claim that, if $r \in U_a \setminus \{0\}$ and $s \in U_b \setminus \{0\}$ are such that $rs = 0$, then $r = a_0$ and $s = b_0$. Let us prove this claim. Assume $r = \sum_{j=1}^{n} a_{i_j}$ and $s = \sum_{l=1}^{m} b_{k_l}$ with $0 \leq i_1 < i_2 < \cdots < i_n$ and $0 \leq k_1 < k_2 < \cdots < k_m$. Assume $a_{i_n} b_{k_m} \neq 0$. Since $rs = 0$, we have to cancel $a_{i_n} b_{k_m}$ with another monomial of the form $a_{i_j} b_{k_l}$, but, since the relations in (a) preserve the length of the elements of R, $l(a_{i_n} b_{k_m}) = i_n + k_m > l(rs - a_{i_n} b_{k_m})$, which is a contradiction. Therefore, $a_{i_n} b_{k_m} = 0$, so $i_n = k_m = 0$, and then, $r = a_0$ and $s = b_0$.

Now, since $f(x)g(x) = 0$, by defining $r_i = s_j = 0$ for all $i > n$ and $j > m$, we have that $c_k = \sum_{i=0}^{k} r_i s_{k-i} = 0$, for all $k \in \{0, \ldots, n+m\}$. In particular $r_0 s_0 = 0$, so, by the claim, $r_0 = a_0$ and $s_0 = b_0$.

Let $I_0 = \{0 \leq i \leq n \mid r_i \neq 0\} \supseteq \{0, n\}$ and $J_0 = \{0 \leq j \leq m \mid s_j \neq 0\} \supseteq \{0, m\}$. We know that $l(r_i), l(s_j) \geq 0$ for all $i \in I_0$ and for all $j \in J_0$. Assume $l(r_i) = 0$ for all $i \in I_0$, then $r_i = a_0$ for all $i \in I_0$ and

$$c_k = \sum_{i=0}^{k} a_0 s_{k-i} = a_0 (\sum_{j=0}^{k} s_j) = 0$$

for all $k \geq 0$. By the claim, $\sum_{j=0}^{k} s_j = b_0$ for all $k \geq 0$, so $f(x) = \sum_{i \in I_0} a_0 x^i$ and $g(x) = b_0$. Therefore, $r_i s_j = 0$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$. Analogously, if $l(s_j) = 0$ for all $j \in J_0$, we have that $r_i s_j = 0$ for all $0 \leq i \leq n$ and for all $0 \leq j \leq m$.

Assume now that there exist $i \in I_0 \setminus \{0\}$ and $j \in J_0 \setminus \{0\}$ such that $l(r_i), l(s_j) > 0$. We shall see that this is impossible. Let $i_1 > 0$ and $j_1 > 0$ be such that $l(r_i) \leq l(r_{i_1})$ for all $i \in I_0$ and $l(s_j) \leq l(s_{j_1})$ for all $j \in J_0$, and i_1, j_1 are the minimum with this property. Now, since $c_{i_1+j_1} = r_0 s_{i_1+j_1} + \cdots + r_{i_1} s_{j_1} + \cdots + r_{i_1+j_1} s_0 = 0$ and $r_{i_1} s_{j_1} \neq 0$, we need to cancel the monomial $\delta(r_{i_1} s_{j_1})$ in this expression. By the definition of i_1 and j_1, we have that $l(r_{i_1} s_{j_1}) = l(\delta(r_{i_1} s_{j_1})) \geq l(r_k s_{i_1+j_1-k})$ for all $1 \leq k \leq i_1 + j_1$. Assume that there exists $k \neq i_1$ such that $l(r_{i_1} s_{j_1}) = l(r_{i_1}) + l(s_{j_1}) = l(r_k s_{i_1+j_1-k}) = l(r_k) + l(s_{i_1+j_1-k})$. Then, $l(r_{i_1}) = l(r_k)$ and $l(s_{j_1}) = l(s_{i_1+j_1-k})$. By minimality of i_1, we have that $k > i_1$, so $i_1 + j_1 - k < j_1$, which is a contradiction with the minimality of j_1.

Therefore, R is an Armendariz ring. ∎

Let α be an automorphism of R. Let Δ^* be the set of all left annihilators of $R[[x;\alpha]]$, $\Delta^* = \{l.ann_{R[[x;\alpha]]}(V) \mid V \subseteq R[[x;\alpha]]\}$ and recall that $\Gamma = \{l.ann_R(U) \mid U \subseteq R\}$.

If $U \subseteq R$, then $R[[x;\alpha]]l.ann_R(U) = l.ann_{R[[x;\alpha]]}(U)$, and, if $V \subseteq R[[x;\alpha]]$, we have that $l.ann_{R[[x;\alpha]]}(V) \cap R = l.ann_R(C_V)$, where $C_V = \bigcup_{f(x)\in V} C_f$ and, if $f(x) = \sum_{i\geq 0} a_i x^i$, $C_f = \{a_0, a_1, \dots, a_n, \dots\} \cup \{0\}$. Therefore, we can define the maps $\phi^* : \Gamma \to \Delta^*$ by $\phi^*(A) = R[[x;\alpha]]A$ for all $A \in \Gamma$, and $\psi^* : \Delta^* \to \Gamma$ by $\psi^*(B) = B \cap R$ for all $B \in \Delta^*$. It is easy to see that ϕ^* is injective and that ψ^* is surjective. Moreover, ϕ^* is bijective if and only if ψ^* is bijective, and, in this case, one is the inverse of the other. Cortes in [3, Lemma 2.7] proved that ϕ^* is bijective if and only if R is strongly α-skew Armendariz.

Cortes also proved that, although the zip property is not hereditary, when the skew power series ring of a ring R is left zip, the ring R itself is left zip ([3, Theorem 2.8]). Someone could think that, analogously as we did in Lemma 2.2, a similar result for rings with the Beachy-Blair condition can be proven by requiring the ring to be α-compatible. However, this is not true, as we will see in Section 4 (see Proposition 4.4).

Again Cortes in [3, Theorem 2.8] proved that, when the ring R is strongly α-skew Armendariz, R is left zip if and only if $R[[x;\alpha]]$ is left zip. A similar result is also true for rings with the Beachy-Blair condition.

Theorem 3.5

Let R be a strongly α-skew Armendariz ring, with α an automorphism of R. If R satisfies the left Beachy-Blair condition, then $R[[x;\alpha]]$ satisfies the left Beachy-Blair condition.

Proof. Denote by S the skew power series ring, $S = R[[x;\alpha]]$. Suppose that R satisfies the Beachy-Blair condition. Let $J \subseteq S$ be a left ideal of S such that $l.ann_S(J) = \{0\}$. Then, if $J' = JS$ is the (two-sided) ideal generated by J, we have that $l.ann_S(J') = \{0\}$. Then, $l.ann_R(C_{J'}) = \psi^*(l.ann_S(J')) = \psi^*(\{0\}) = \{0\}$ and $C_{J'} \subseteq R$. We shall see that $C_{J'}$ is an ideal of R.

(a) If $s \in C_{J'}$, then there exists $f(x) = \sum_{i\geq 0} a_i x^i \in J'$ such that $s = a_i$ for some i. Let $r \in R$. Since J' is an ideal of S, $g(x) = rf(x)$, $h(x) = f(x)\alpha^{-i}(r) \in J'$, and the coefficients of x^i in $g(x)$ and $h(x)$ are ra_i and $a_i r$ respectively, so $rs, sr \in C_{J'}$.

(b) If $r_1, r_2 \in C_{J'}$, there exist $f(x), g(x) \in J'$ such that $f(x) = \sum_{i\geq 0} a_i x^i$, $g(x) = \sum_{j\geq 0} b_j x^j$ and $r_1 = a_i$, $r_2 = b_j$ for some $i, j \geq 0$. We want to see that $r_1 + r_2 \in C_{J'}$. Assume without loss of generality that $i \leq j$. Since $f(x), g(x) \in J'$ and J' is an ideal of S, we have that $h(x) = f(x)x^{j-i} + g(x) \in J'$, and the coefficient in x^j of $h(x)$ is $a_i + b_j$, so $r_1 + r_2 \in C_{J'}$.

Now, since R satisfies the left Beachy-Blair condition, there exists a finite subset $X \subseteq C_{J'}$ such that $l.ann_R(X) = \{0\}$. Assume that $X = \{a_1, \ldots, a_n\}$, then, for every a_i there exists a power series $f_i(x) \in J'$ such that $a_i \in C_{f_i}$. Let $Y = \{f_1(x), \ldots, f_n(x)\} \subseteq J'$. Clearly $X \subseteq C_Y$, so $l.ann_R(C_Y) \subseteq l.ann_R(X) = \{0\}$. Since R is strongly α-skew Armendariz and

$$l.ann_R(C_Y) = \psi^*(l.ann_S(Y)) = \{0\},$$

by [3, Lemma 2.7], we have that $l.ann_S(Y) = \{0\}$.

By the definition of J', there exist integers $m_1, \ldots, m_n \geq 0$ and power series $f_{i,j}(x) \in J$, $s_{i,j}(x) \in S$ for all $1 \leq i \leq n$, $0 \leq j \leq m_i$ such that

$$f_i(x) = \sum_{j=0}^{m_i} f_{i,j}(x) s_{i,j}(x).$$

Let $Y' = \{f_{i,j}(x) \mid 1 \leq i \leq n \text{ and } 0 \leq j \leq m_i\} \subseteq J$. Clearly, $l.ann_S(Y') \subseteq l.ann_S(Y) = \{0\}$. Therefore, S satisfies the left Beachy-Blair condition. ∎

An easy consequence of Theorem 3.5 is the following:

Corollary 3.6

Let R be a strongly Armendariz ring. If R satisfies the left Beachy-Blair condition then $R[[x]]$ satisfies the left Beachy-Blair condition.

Although, by Example 3.1, there exists a right zip ring R such that $R[[x]]$ is not right zip, it is still an open problem whether there exists such an example for the Beachy-Blair condition or not. However, it would be really surprising that this example did not exist.

4 Examples

In this section we give a negative answer to some interesting questions about the behavior of the zip property and the Beachy-Blair condition. These questions are the following:

(Q1) Let R be a ring. Does R being a left zip ring implies R to be a right zip ring?

(Q2) Let R be a ring and α an automorphism of R. Does $R[[x;\alpha]]$ satisfying the left Beachy-Blair condition implies R to satisfy the left Beachy-Blair condition? What if R is α-compatible?

Handelman and Lawrence gave an example of a prime ring in which every (faithful) left ideal is cofaithful but which does not have the analogous property for right ideals (see [6, Example 1]). Therefore, there exist examples of rings satisfying the left Beachy-Blair condition that do not satisfy the right Beachy-Blair condition. However, for the zip property, there are not explicit examples

in the existing literature answering (Q1). We will see in Proposition 4.5 that the answer to (Q1) is negative.

In Lemma 2.2, we proved that if the polynomial extension of a ring that is α-compatible, satisfies the left Beachy-Blair condition, then the ring itself satisfies the left Beachy-Blair condition, and Cortes in [3, Theorem 2.8] proved that if $R[[x,\alpha]]$ is left zip, then R is left zip as well. However, the answer to (Q2) is negative, as we will see in Proposition 4.4.

Let $\mathbb{K}$ be a field, $A = \{a_i \mid i \geq 0\}$ and $B = \{b_j \mid j \geq 0\}$. Let R be the $\mathbb{K}$-algebra presented with set of generators $A \cup B$ and with relations:

(c1) $b_j b_l = a_i b_j = 0$ for all $i, j, l \geq 0$,

(c2) $b_j a_i = 0$ if and only if $j \geq i$.

We denote by $\langle U \rangle$, with $U \subseteq R$, the multiplicative subsemigroup generated by the elements in U. Let $R_A = \mathbb{K} + \mathbb{K}[\langle A \rangle]$. Note that $R_A \subseteq R$ is an integral domain.

Let $V = \{b_j a_{i_1} \cdots a_{i_n} \mid n \geq 1, i_1, \ldots, i_n \geq 0 \text{ and } j < i_1\}$. Let U_a, U_b and U_{ba} be the $\mathbb{K}$-linear span of $\langle A \rangle$, B and V respectively. Clearly, $\mathfrak{B} = \{1\} \cup \langle A \rangle \cup B \cup V$ is a $\mathbb{K}$-basis of R, and, for every $r \in R$, there exist unique $r_0 \in \mathbb{K}$, $r_a \in U_a$, $r_b \in U_b$ and $r_{ba} \in U_{ba}$ such that $r = r_0 + r_a + r_b + r_{ba}$. For all $r \in R$, $r = \sum_{z \in \mathfrak{B}} r(z)z$, where $r(z) \in \mathbb{K}$ and $r(z) = 0$ for almost all $z \in \mathfrak{B}$. We define the support of r, $Supp(r)$, to be $Supp(r) = \{z \in \mathfrak{B} \mid r(z) \neq 0\}$.

Let $S = R[[x]]$. We denote by $U[[x]]$, with $U \subseteq R$, the set of all power series in S with coefficients in U. For all $f(x) \in S$, there exist unique $f_0(x) \in \mathbb{K}[[x]]$, $f_a(x) \in U_a[[x]]$, $f_b(x) \in U_b[[x]]$ and $f_{ba}(x) \in U_{ba}[[x]]$ such that $f(x) = f_0(x) + f_a(x) + f_b(x) + f_{ba}(x)$. Note that $R_A[[x]] \subseteq R[[x]]$ is an integral domain and $S = R_A[[x]] \oplus BS$, so, for every $f(x) \in S$, there exist unique $f_A(x) \in R_A[[x]]$ and $f_B(x) \in BS$ such that $f(x) = f_A(x) + f_B(x)$. It is clear that $f_A(x) = f_0(x) + f_a(x)$ and $f_B(x) = f_b(x) + f_{ba}(x)$.

Lemma 4.1

(a) Let $r \in U_{ba}$ and $s \in \mathbb{K} \oplus U_a$. If r, s are non-zero then $rs \neq 0$.

(b) $l.ann_R(A) = \{0\}$.

(c) Let $r = r_0 + r_a + r_b + r_{ba} \in R$, with $r_0 \in \mathbb{K}$, $r_a \in U_a$, $r_b \in U_b$ and $r_{ba} \in U_{ba}$. If r_0 is non-zero then $l.ann_R(r) = r.ann_R(r) = \{0\}$.

Proof.

(a) Let $r \in U_{ba} \setminus \{0\}$ and $s \in \mathbb{K} \oplus U_a \setminus \{0\}$. Then, there exist $w_1 = b_j a_{i_1} \cdots a_{i_n} \in Supp(r)$ and $w_2 = a_{k_1} \cdots a_{k_m} \in Supp(s)$ such that the total degree in all the generators in A is maximum, or $w_2 = 1$ if $s \in \mathbb{K}$. It is clear that if $s \in \mathbb{K} \setminus \{0\}$, then $rs \neq 0$, so we may assume that $s \notin \mathbb{K}$.

Assume $rs = 0$. Since $b_j a_{i_1} \cdots a_{i_n} a_{k_1} \cdots a_{k_m} \neq 0$, there exist $w_1' = b_j a_{i_1'} \cdots a_{i_{n'}'} \in Supp(r)$ and $w_2' \in Supp(s)$ such that

$$b_j a_{i_1} \cdots a_{i_n} a_{k_1} \cdots a_{k_m} = b_j a_{i_1'} \cdots a_{i_{n'}'} w_2'$$

and $(w_1, w_2) \neq (w_1', w_2')$.

By the choice of w_1 and w_2 we have that $n \geq n'$ and $m \geq deg_A(w_2')$, where $deg_A(-)$ denotes the total degree in all the generators in A. Thus, $n = n'$ and $deg_A(w_2') = m$, so $i_p = i_p'$ and $w_2' = w_2$ for all $1 \leq p \leq n$, and then $(w_1, w_2) = (w_1', w_2')$, which is a contradiction. Therefore, $rs \neq 0$.

(b) Let $r \in l.ann_R(A)$. Assume $r = r_0 + r_a + r_b + r_{ba}$ with $r_0 \in \mathbb{K}$, $r_a \in U_a$, $r_b \in U_b$ and $r_{ba} \in U_{ba}$. We have that $ra_i = r_0 a_i + r_a a_i + r_b a_i + r_{ba} a_i = 0$ for all $i \geq 0$, so $(r_0 + r_a)a_i = 0$ and $(r_b + r_{ba})a_i = 0$ for all $i \geq 0$. Since R_A is an integral domain, we have that $r_0 + r_a = 0$, so $r_0 = r_a = 0$.

Now we have that $r_b a_i = -r_{ba} a_i$ for all $i \geq 0$. Suppose that $r_b a_i \neq 0$ for some $i \geq 0$. Then, there exist $b_l a_i \in Supp(r_b a_i)$ and $w a_i \in Supp(r_{ba} a_i)$, with $w \in Supp(r_{ba})$, such that $b_l a_i = w a_i \neq 0$, but this is impossible since $deg_A(b_l a_i) = 1$ and $deg_A(w a_i) \geq 2$, and the defining relations (c1) and (c2) preserve degrees in all the generators in A. Therefore, $r_b a_i = r_{ba} a_i = 0$ for all $i \geq 0$. By (a), since $r_{ba} a_0 = 0$ and $a_0 \neq 0$, we have that $r_{ba} = 0$. Finally, we shall see that $r_b = 0$.

Suppose $r_b \neq 0$ and let $r_b = \displaystyle\sum_{i=1}^{n} \lambda_i b_{j_i}$ with $\lambda_i \in \mathbb{K} \setminus \{0\}$ and $j_1 < j_2 < \cdots < j_n$. Then, by (c2), $r_b a_{j_1+1} = \lambda_1 b_{j_1} a_{j_1+1} \neq 0$, which is a contradiction. Therefore, $r_b = 0$ and then $l.ann_R(A) = \{0\}$.

(c) Let $r = r_0 + r_a + r_b + r_{ba} \in R$ with $r_0 \in \mathbb{K}$, $r_a \in U_a$, $r_b \in U_b$ and $r_{ba} \in U_{ba}$ be such that $r_0 \neq 0$. Let $s = s_0 + s_a + s_b + s_{ba} \in l.ann_R(r)$ with $s_0 \in \mathbb{K}$, $s_a \in U_a$, $s_b \in U_b$ and $s_{ba} \in U_{ba}$. Then, since $sr = 0$, we have that $s_0 r_0 = 0$, so $s_0 = 0$, and, by (c1) and (c2), $sr = s_a r_0 + s_a r_a + s_b r_0 + s_b r_a + s_{ba} r_0 + s_{ba} r_a$. Now we have that $s_a r_a + s_a r_0 = 0$, $s_b r_0 = 0$, so $s_b = 0$, and $s_{ba} r_0 + s_{ba} r_a = 0$.

We have that $s_a(r_0 + r_a) = 0$ so, since R_A is an integral domain and $(r_0 + r_a) \neq 0$, then $s_a = 0$. We also have that $s_{ba} r_0 = -s_{ba} r_a$. If $s_{ba} \neq 0$, then $r_a \neq 0$ and $deg_A(s_{ba} r_0) = deg_A(s_{ba}) = deg_A(s_{ba} r_a) = deg_A(s_{ba}) + deg_A(r_a) > deg_A(s_{ba})$, which is a contradiction. Therefore $s = s_{ba} = 0$, so $l.ann_R(r) = \{0\}$.

Let $s = s_0 + s_a + s_b + s_{ba} \in r.ann_R(r)$ with $s_0 \in \mathbb{K}$, $s_a \in U_a$, $s_b \in U_b$ and $s_{ba} \in U_{ba}$. Then, since $rs = 0$ we have that $r_0 s_0 = 0$, so $s_0 = 0$, and, by (c1), $rs = r_0 s_a + r_a s_a + r_0 s_b + r_0 s_{ba} + r_b s_a + r_{ba} s_a$. Now we have that $(r_0 + r_a)s_a = 0$, $r_0 s_b = 0$, so $s_b = 0$, and $r_0 s_{ba} + r_b s_a + r_{ba} s_a = 0$.

Since $(r_0 + r_a)s_a = 0$, R_A is an integral domain and $(r_0 + r_a) \neq 0$, we have that $s_a = 0$. Finally, we have that $r_0 s_{ba} = 0$, so $s_{ba} = 0$ and then, $s = 0$. Therefore, $r.ann_R(r) = \{0\}$.

∎

Lemma 4.2

Let $r_1, \ldots, r_n \in R \setminus \{0\}$ and $s \in U_a \setminus \{0\}$. Then, $\sum_{i=1}^{n} r_i a_i s = 0$ if and only if $r_i a_i s = 0$ for all $1 \leq i \leq n$.

Proof. Let $r_1, \ldots, r_n \in R \setminus \{0\}$ and $s \in U_a \setminus \{0\}$ be such that $\sum_{i=1}^{n} r_i a_i s = 0$.

Suppose that there exists $1 \leq i \leq n$ such that $deg_A(r_i) > 0$. For each $1 \leq i \leq n$, let $w_i \in Supp(r_i)$ be such that $deg_A(w_i)$ is maximum. Let i_0 be such that $deg_A(w_{i_0}) = \max\{deg_A(w_i) \mid 1 \leq i \leq n\}$. Let $u \in Supp(s)$ be such that $deg_A(u)$ is maximum. Then, $0 \neq w_{i_0} a_{i_0} u \in Supp(r_{i_0} a_{i_0} s)$. Since $\sum_{i=1}^{n} r_i a_i s = 0$, there exist $w \in \bigcup_{i=1}^{n} Supp(r_i)$, $u' \in Supp(s)$ and $1 \leq i_1 \leq n$ such that $(w, i_1, u') \neq (w_{i_0}, i_0, u)$ and $w a_{i_1} u' = w_{i_0} a_{i_0} u \neq 0$. Then, $deg_A(w a_{i_1} u') = deg_A(w) + deg_A(u') + 1 = deg_A(w_{i_0} a_{i_0} u) = deg_A(w_{i_0}) + deg_A(u) + 1$, so, by the definition of w_{i_0} and u, we have that $deg_A(w) = deg_A(w_{i_0})$ and $deg_A(u) = deg_A(u')$. Thus, $w = w_{i_0}$, $u = u'$ and $i_1 = i_0$, which is a contradiction. Therefore, for all $1 \leq i \leq n$, $deg_A(r_i) = 0$, so $r_i \in U_b \oplus \mathbb{K}$, $r_i \neq 0$, for all $1 \leq i \leq n$. Similarly, it is easy to see that $1 \notin Supp(r_i)$ for all $1 \leq i \leq n$.

Then we have $r_i \in U_b$ for all $1 \leq i \leq n$. Let $u \in Supp(s)$ be such that $deg_A(u)$ is maximum. Assume that there exists i such that $r_i a_i s \neq 0$. Then, there exists $b_j \in Supp(r_i)$ such that $b_j a_i u \in Supp(r_i a_i s)$. Since $\sum_{i=1}^{n} r_i a_i s = 0$, there exist $1 \leq i_1 \leq n$, $b_l \in Supp(r_{i_1})$ and $u' \in Supp(s)$ such that $(j, i, u) \neq (l, i_1, u')$ and $b_j a_i u = b_l a_{i_1} u' \neq 0$, but this is impossible. Therefore, $r_i a_i s = 0$ for all $1 \leq i \leq n$. ∎

Lemma 4.3

Let $f(x), g(x) \in R[[x]] \setminus \{0\}$. If $f(x)g(x) = 0$ then $f_0(x) = g_0(x) = 0$, where $f(x) = f_0(x) + f_a(x) + f_b(x) + f_{ba}(x)$, $g(x) = g_0(x) + g_a(x) + g_b(x) + g_{ba}(x)$, with $f_0(x), g_0(x) \in \mathbb{K}[[x]]$, $f_a(x), g_a(x) \in U_a[[x]]$, $f_b(x), g_b(x) \in U_b[[x]]$ and $f_{ba}(x), g_{ba}(x) \in U_{ba}[[x]]$.

Proof. Let $f(x) = f_0(x)+f_a(x)+f_b(x)+f_{ba}(x)$, $g(x) = g_0(x)+g_a(x)+g_b(x)+g_{ba}(x) \in R[[x]] \setminus \{0\}$ be such that $f(x)g(x) = 0$. Let $f_A(x) = f_0(x) + f_a(x)$, $g_A(x) = g_0(x) + g_a(x)$ and $f_B(x) = f_b(x) + f_{ba}(x)$, $g_B(x) = g_b(x) + g_{ba}(x)$. Then, we have that $f_0(x)g_0(x) = 0$ and $f_A(x)g_A(x) = 0$. Since $\mathbb{K}[[x]]$ and $R_A[[x]]$ are integral domains, we have that either $f_A(x) = 0$ or $g_A(x) = 0$ and either $f_0(x) = 0$ or $g_0(x) = 0$.

Assume $f_0(x) \neq 0$. Then, $g_A(x) = 0$ and $f(x)g(x) = f_0(x)g_B(x) = 0$. Let $f_0(x) = \sum_{i \geq i_0} \varepsilon_i x^i$, with $\varepsilon_i \in \mathbb{K}$ for all $i \geq i_0$ and $\varepsilon_{i_0} \neq 0$. Since $g(x) = g_B(x) \neq 0$, we have that $g_B(x) = \sum_{j \geq j_0} r_j x^j$, with $r_j \in BR$ for all $j \geq j_0$ and $r_{j_0} \neq 0$.

The coefficient of $x^{i_0+j_0}$ in $f_0(x)g_B(x)$ is $\varepsilon_{i_0}r_{j_0} \neq 0$, which is a contradiction. Therefore, $f_0(x) = 0$.

Assume now that $g_0(x) \neq 0$. Then, $f_A(x) = 0$ and $f(x)g(x) = f_B(x)g_A(x) = f_B(x)g_0(x) + f_B(x)g_a(x)$. Therefore, $f_b(x)g_0(x) = 0$ and

$$f_{ba}(x)g_0(x) = -f_B(x)g_a(x).$$

Let $g_0(x) = \sum_{i\geq i_0}\varepsilon_i x^i$, with $\varepsilon_i \in \mathbb{K}$ for all $i \geq i_0$ and $\varepsilon_{i_0} \neq 0$. If $f_b(x) \neq 0$, then $f_b(x) = \sum_{j\geq j_0} r_j x^j$ for some $r_j \in U_b$ for all $j \geq j_0$ and $r_{j_0} \neq 0$. Now, the coefficient of $x^{i_0+j_0}$ in $f_b(x)g_0(x)$ is $r_{j_0}\varepsilon_{i_0} \neq 0$, which is a contradiction. Therefore, $f_b(x) = 0$ and $f_{ba}(x)g_A(x) = 0$. Since $f(x) = f_{ba}(x) \neq 0$, $f_{ba}(x) = \sum_{j\geq j_1} s_j x^j$, with $s_j \in U_{ba}$ for all $j \geq j_1$ and $s_{j_1} \neq 0$. Since $g_A(x) \neq 0$, $g_A(x) = \sum_{i\geq i_1} t_i x^i$, with $t_i \in \mathbb{K}\oplus U_a$ and $t_{i_1} \neq 0$. Now the coefficient of $x^{i_1+j_1}$ in $f_{ba}(x)g_A(x)$ is $s_{j_1}t_{i_1} = 0$, which is a contradiction to (a) in Lemma 4.1. Therefore, $g_0(x) = f_0(x) = 0$. ∎

Proposition 4.4

The ring R does not satisfy the left Beachy-Blair condition. However, $R[[x]]$ satisfies the left Beachy-Blair condition.

Proof. Let I be the left ideal of R generated by A. By (b) in Lemma 4.1, $l.ann_R(A) = \{0\}$, so $l.ann_R(I) = \{0\}$. Let $F = \{r_1, \ldots, r_n\}$ be a finite subset of I. Then, there exist $m_1, \ldots, m_n \geq 1$, $w_{i,j} \in \langle A\rangle \cup V$ (recall that $V = \{b_j a_{i_1}\cdots a_{i_n} \mid n \geq 1, i_1, \ldots, i_n \geq 0 \text{ and } j < i_1\}$) and $\lambda_{i,j} \in \mathbb{K}\setminus\{0\}$ for all $1 \leq j \leq m_i$ and for all $1 \leq i \leq n$ such that $r_i = \sum_{j=1}^{m_i}\lambda_{i,j}w_{i,j}$ and $w_{i,j} \neq w_{i,l}$ for all $1 \leq i \leq n$ and for all $j \neq l$. Let $X = \{a_{k_{i,j}} \mid w_{i,j} = a_{k_{i,j}}w'_{i,j}$ for some $1 \leq i \leq n, 1 \leq j \leq m_i$ and $w'_{i,j} \in \langle A\rangle \cup \{1\}\}$. If $X = \emptyset$, then $b_0 r_i = 0$ for all $1 \leq i \leq n$, so $b_0 \in l.ann_R(F)$. Assume $X \neq \emptyset$ and let $k = \max\{k_{i,j} \mid a_{k_{i,j}} \in X\}$. Then, by the defining relations (c1) and (c2), $b_k r_i = 0$ for all $1 \leq i \leq n$, so $b_k \in l.ann_R(F)$. Therefore, R does not satisfy the left Beachy-Blair condition.

We shall see now that $S = R[[x]]$ satisfies the left Beachy-Blair condition. Let J be a left ideal of S such that $l.ann_S(J) = \{0\}$. If there exists $f(x) \in J$ such that $f_0(x) \neq 0$, then, by Lemma 4.3, we have that $l.ann_S(f(x)) = \{0\}$.

Suppose that for all $f(x) \in J$, $f_0(x) = 0$ and let $g(x) = \sum_{i\geq 0} a_i x^i$. Since $l.ann_S(J) = \{0\}$, there exists $f(x) \in J$ such that $g(x)f(x) \neq 0$, and $h(x) = g(x)f(x) = g(x)f_a(x) \in J$, since J is a left ideal of S. We shall see that $l.ann_S(h(x)) = \{0\}$.

Suppose that there exists $t(x) \in S\setminus\{0\}$ such that $t(x)h(x) = 0$. Since $h(x) \in R_A[[x]]\setminus\{0\}$, it is clear by the proof of Lemma 4.3, that $t(x) = t_b(x) + t_{ba}(x)$

$(t_A(x) = 0,\ t(x) = t_B(x))$. Let $f_a(x) = \sum_{i\geq i_0} r_i x^i$, where $r_i \in U_a$ and $r_{i_0} \neq 0$. Define $c_k = \sum_{j=i_0}^{k} a_{k-j} r_j \in U_a$ for all $k \geq i_0$, then, $h(x) = \sum_{k\geq i_0} c_k x^k$. Suppose that $t(x) = \sum_{j\geq 0} s_j x^j$. We may assume without loss of generality that $s_0 \neq 0$ and $s_j = s_{j,b} + s_{j,ba}$, where $s_{j,b} \in U_b$ and $s_{j,ba} \in U_{ba}$, for all $j \geq 0$. Then, for all $k \geq i_0$, we have that

$$\sum_{j=i_0}^{k} s_{k-j} c_j = 0.$$

For $k = i_0$ we have that $s_0 c_{i_0} = s_0 a_0 r_{i_0} = s_{0,ba} a_0 r_{i_0} = 0$, because, by the relation (c2), $s_{0,b} a_0 = 0$. Then, by (a) in Lemma 4.1, $s_{0,ba} = 0$, so $s_0 = s_{0,b} \neq 0$.

For $k = i_0+1$ we have that $s_0 c_{i_0+1} + s_1 c_{i_0} = s_{0,b}(a_0 r_{i_0+1} + a_1 r_{i_0}) + s_1 a_0 r_{i_0} = s_{0,b} a_1 r_{i_0} + s_{1,ba} a_0 r_{i_0} = 0$, since by the relation (c2), $s_{0,b} a_0 = s_{1,b} a_0 = 0$. Then, by Lemma 4.2, $s_{0,b} a_1 r_{i_0} = s_{1,ba} a_0 r_{i_0} = 0$, so, by (a) in Lemma 4.1, $s_{0,b} a_1 = 0$ and $s_{1,ba} = 0$.

Induction Hypothesis: Assume that $s_{i,ba} = 0$, for all $i < n$, and $s_{j,b} a_{i-j} = 0$, for all $i < n$ and for all $0 \leq j \leq i$.

We shall see that $s_{n,ba} = 0$ and that $s_{0,b} a_n = s_{1,b} a_{n-1} = \cdots = s_{n,b} a_0 = 0$.

For $k = n + i_0$ we have that $\sum_{j=i_0}^{n+i_0} s_{n+i_0-j} c_j = \sum_{j=0}^{n} s_{n-j} c_{j+i_0} = 0$. By the induction hypothesis, we have that, for all $j > 0$,

$$s_{n-j} c_{j+i_0} = s_{n-j,b} c_{j+i_0} = \sum_{k=0}^{j} s_{n-j,b} a_{j-k} r_{k+i_0} = s_{n-j,b} a_j r_{i_0}.$$

Thus, $\sum_{j=0}^{n} s_{n-j} c_{j+i_0} = \sum_{j=0}^{n} s_{n-j} a_j r_{i_0} = 0$, so, by Lemma 4.2, we have that $s_{n-j} a_j r_{i_0} = 0$ for all $0 \leq j \leq n$. In particular, by (c2), $s_n a_0 r_{i_0} = s_{n,ba} a_0 r_{i_0} = 0$, so, by (a) in Lemma 4.1, $s_{n,ba} = 0$. Moreover, by (a) in Lemma 4.1, since $s_{n-j} a_j r_{i_0} = s_{n-j,b} a_j r_{i_0} = 0$ for all $j > 0$, we have that $s_{n-j,b} a_j = 0$ for all $j > 0$ and $s_{n,b} a_0 = 0$ by the relation (c2).

Therefore, $s_j = s_{j,b}$ for all $j \geq 0$ and $s_j a_{i-j} = 0$ for all $i \geq 0$ and for all $0 \leq j \leq i$. In particular, $s_0 a_i = 0$ for all $i \geq 0$, so $s_0 \in l.ann_R(A)$, but, by (b) in Lemma 4.1, $l.ann_R(A) = \{0\}$, which is a contradiction. Thus, $l.ann_S(h(x)) = \{0\}$.

Therefore, $R[[x]]$ satisfies the left Beachy-Blair condition. ∎

Note that every ring is 1_R-compatible, where 1_R denotes the identity automorphism of R, and $R[[x; 1_R]] = R[[x]]$. Therefore, Proposition 4.4 answers (Q2) in the negative even in the case of α-compatible rings.

Proposition 4.5

The ring R is right but not left zip.

Proof. By Proposition 4.4, we know that R does not satisfy the left Beachy-Blair condition. Therefore, R is not left zip.

Let $X \subseteq R$ be such that $r.ann_R(X) = \{0\}$. Suppose that $r_0 = 0$ for all $r = r_0 + r_a + r_b + r_b a \in X$. Then, $rb_0 = r_a b_0 + r_b b_0 + r_{ba} b_0 = 0$ by the relation (c1), for all $r \in X$, so $b_0 \in r.ann_R(X)$, which is a contradiction. Therefore, there exists $r = r_0 + r_a + r_b + r_b a \in X$ such that $r_0 \neq 0$. Then, by (c) in Lemma 4.1, $r.ann_R(r) = \{0\}$, so R is a right zip ring. ∎

Note that $R[[x]]$ satisfies the left Beachy-Blair condition but it is not left zip, since, by [3, Theorem 2.8], if $R[[x]]$ is left zip then R is left zip as well, but we have seen that R does not satisfy the left Beachy-Blair condition and, therefore, is not left zip.

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